

Sharp Bounds for the Complexity of Semi-Equitable Coloring of Cubic and Subcubic Graphs*

Hanna Furmańczyk[†] Marek Kubale[‡]

Abstract

In this note we consider the complexity of semi-equitable k -coloring, $k \geq 4$, of the vertices of a cubic or subcubic graph G . In particular, we show that, given n -vertex subcubic graph G and $k \geq 4$, a semi-equitable k -coloring of G is NP-hard if $s \geq 7n/20$, and polynomially solvable if $s \leq 7n/21$, where s is the size of maximum color class of the coloring.

1 Introduction

We say that a graph $G = (V, E)$ is *equitably k -colorable* if and only if its vertex set can be partitioned into independent sets $V_1, \dots, V_k \subset V$ such that $|V_i| - |V_j| \in \{-1, 0, 1\}$ for all $i, j = 1, \dots, k$. The smallest k for which G admits such a coloring is called the *equitable chromatic number* of G and denoted $\chi_=(G)$. Graph G has a *semi-equitable coloring*, if there exists a partition of its vertices into independent sets $V_1, \dots, V_k \subset V$ such that one of these subsets, say V_1 is of size $s \notin \{\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil\}$, and the remaining subgraph $G - V_1$ is equitably $(k-1)$ -colorable. These two non-classical models of graph coloring have potential applications in multiprocessor scheduling of unit-execution time jobs [6, 7].

In the following we will say that graph G has (V_1, \dots, V_k) coloring to express explicitly a partition of V into k independent sets. If, however, only cardinalities of color classes are important, we will use the notation $[|V_1|, \dots, |V_k|]$.

The following two theorems on equitable graph coloring are well-worth mentioning. First, Hajnal and Szemerédi [2] proved

Theorem 1.1 ([2]). *If G is a graph satisfying $\Delta(G) \leq k$, then G has an equitable $(k+1)$ -coloring.*

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[†]Institute of Informatics, University of Gdańsk, Wita Stwosza 57, 80-308 Gdańsk, Poland. e-mail: hanna@inf.ug.edu.pl

[‡]Department of Algorithms and System Modelling, Gdańsk University of Technology, Narutowicza 11/12, 80-233 Gdańsk, Poland. e-mail: kubale@eti.pg.gda.pl

This theorem implies that every cubic graph has an equitable k -coloring for every $k \geq 4$. Kierstead et al. [9] gave a simple algorithm for obtaining such a coloring in $O(n^2)$ time. Secondly, Chen et al. [1] proved

Theorem 1.2 ([1]). *If G is a connected 3-chromatic cubic graph, then there exists an equitable 3-coloring of G .*

The problem of semi-equitable 3-coloring of connected cubic graphs was introduced in [4]. In this note we extend those results to an arbitrary number $k \geq 4$ of colors and to, possibly disconnected, subcubic graphs, where by a *subcubic graph* we mean a graph G whose vertex degrees fulfill $\deg(v) \leq 3$ for all $v \in V$.

The remaining of the paper is organized as follows. In order to develop a polynomial time algorithm for semi-equitable k -coloring of subcubic graphs, we generalize Chen et al.'s theorem in Section 2. In Section 3 we show that, in contrast to equitable coloring, the problem of semi-equitable coloring becomes NP-hard already if $k = 4$. More precisely, we show that computing a semi-equitable k -coloring of a subcubic graph whose maximum color class is of size at least $7n/20$ is NP-hard. Finally, in Section 4 we show how to obtain in $O(n^2)$ time a semi-equitable k -coloring of a subcubic graph whose maximum color class is at most $n/3$.

2 Equitable 3-coloring of subcubic graphs

In fact, Chen et al. [1] proved that $\chi(G) = \chi_=(G)$ for any connected cubic graph G . Their proof starts from any proper 3-coloring of G and it relies on successive decreasing the *width* of the coloring, i.e. the difference between the cardinality of the largest and smallest independent set, step by step until a coloring is equitable. Actually, this procedure works for every 3-coloring of any cubic graph, except for $K_{3,3}$.

Corollary 2.1 ([1]). *If G is a connected cubic graph such that $K_{3,3} \neq G \neq K_4$, then there exists an equitable 3-coloring of G .* \square

We claim that their approach can be slightly modified to hold for subcubic graphs, not necessarily connected.

Theorem 2.2. *If G is a subcubic graph, $K_4 \neq G \neq K_{3,3}$, then there exists an equitable 3-coloring of G .*

Before we prove Theorem 2.2, we recall some notation used by Chen et al. [1]. Let AkB denote the set $\{x \in A : x \text{ is adjacent to exactly } k \text{ vertices of } B\}$, while $A \deg B$ denotes the set of all vertices in A having all its neighbors in set B , namely $A \deg B = \{x \in A : x \text{ is adjacent to all } \deg(x) \text{ vertices of } B\}$.

The notation $A \Leftrightarrow B$ means that we exchange the color of vertices in A into the color of vertices in B and vice versa. A one-way arrow $A \Leftarrow B$ means that we change the color of B into the color of A . We write $A \Leftarrow x$ when $B = \{x\}$.

Lemma 2.3 ([1]). *Let $G(X, Y)$ be a connected bipartite subcubic graph such that $|X| = m \geq n = |Y|$. If $|Y3X| = t$ then $m - n \leq t + 1$.* \square

This lemma can be extended to the following one.

Lemma 2.4. *Let $G(X, Y)$ be a connected bipartite subcubic graph such that $|X| = m \geq n = |Y|$ and let k be the maximum degree of vertices in Y , $1 \leq k \leq 3$. If $|YkX| = t$ then $m - n \leq t + 1$. \square*

Proof. Let $|E(G)| = e$ be the number of edges in G . Since G is connected, we have $e \geq m + n - 1$. On the other hand, the number of edges can be bounded from above by $kt + (k - 1)(n - t)$. The largest bound on e in a cubic graph G is $e \leq 3t + 2(n - t)$. We get the thesis by combining these two inequalities. \square

Proof of Theorem 2.2. We start with a proper 3-coloring of G : (A, B, C) -coloring, $|A| \geq |B| \geq |C|$. If the coloring is not equitable we will decrease the width of the coloring by 1 or 2. We repeat the width decreasing procedure until the obtained coloring is equitable.

Let us assume that the 3-coloring is not equitable, this means that $|A| - |C| \geq 2$. We may assume that there is no isolated vertex in G , since if we have equitably k -colored graph G and we add i isolated vertices, then we can always color the isolated vertices in such a way that graph $G \cup N_i$ is equitably k -colored, where N_i is a null graph. Thus, we have $1 \leq \deg(x) \leq 3$ for each $x \in V(G)$. Now, we consider the following steps (cases).

1. $A \deg B \neq \emptyset$

If $x \in A \deg B$ then $C \Leftarrow x$, thus obtaining a new coloring $(A - \{x\}, B, C \cup \{x\})$. Henceforth, we assume that $A \deg B = \emptyset$.

2. $C \deg A = \emptyset$ or $C \deg A \neq \emptyset$, but $C3A = \emptyset$

Consider the bipartite subgraph $G(A, C)$ induced by A and C . Since $|A| \geq |C| + 2$, $A \deg B = \emptyset$ and there is no isolated vertex in A , then there must exist a component $G(A', C')$ of $G(A, C)$ such that $|A'| = |C'| + 1$. $A' \Leftrightarrow C'$ will decrease the width. Henceforth, we assume that $C3A \neq \emptyset$.

3. $B \deg A = \emptyset$ and let $|C \deg B| = t$

Similar to the previous step, there must exist a component $G(A', C')$ of $G(A, C)$ such that $|A'| > |C'|$. We will consider two subcases.

- (a) $|C'3A'| \leq t$

Then $|A'| - |C'| = t'$, where $t' \leq t + 1$, by Lemma 2.4. Choose the subset $S \subseteq C \deg B$ such that $|S| = t' - 1$. Do: $A' \Leftrightarrow C' \cup S$, which decreases the width by 1 (C' and S are disjoint).

- (b) $|C'3A'| \geq t + 1$

Consider $G(B, C)$. We want to do some recolorings resulting in $B \deg A \neq \emptyset$.

- i. $|B| = |C|$
Do: $B \Leftrightarrow C$.

ii. $|B| > |C|$

Since $B \deg A = \emptyset$ and there is no isolated vertex in G , there must exist a component $G(B'', C'')$ of $G(B, C)$ such that $|B''| > |C''|$. Since $|C'' \deg B''| \leq |C \deg B| = t$, we have $|B''| - |C''| = t''$, where $t'' \leq t+1$. Choose a subset $S'' \subseteq C \deg A$ such that $|S''| = t''$. Do: $B'' \Leftrightarrow C'' \cup S''$ (C'' and S'' are disjoint). Note that changing of the colors assigned to vertices in B'' and $C'' \cup S''$, where $|B''| = |C''| + t'' = |C''| + |S''|$, does not result in decreasing the color width, but results in $B \deg A \neq \emptyset$ so that Step 4 or 5 could be applied.

Henceforth, we assume that $B \deg A \neq \emptyset$.

4. $A \deg C \neq \emptyset$

(a) $|A| = |B|$

Do: $C \Leftarrow x$ for any $x \in B \deg A$, which decreases the color width.

(b) $|A| > |B|$

Let $x \in B \deg A$ and $y \in A \deg C$. Do: $B \Leftarrow y$ and $C \Leftarrow x$.

5. $A \deg C = \emptyset$

If $|A| = |B|$, we do as in the previous step: $C \Leftarrow x$, where $x \in B \deg A$. Let $|A| > |B|$. There must exist a component $G(A', B')$ of $G(A, B)$ such that $|A'| > |B'|$.

(a) $B' \deg A' = \emptyset$

Let $x \in B \deg A$. By Lemma 2.4 we have that $|A'| = |B'| + 1$. Do: $C \Leftarrow x$ and $A' \Leftrightarrow B'$.

(b) $B' \deg A' \neq \emptyset$

i. There is some $x \in B' \deg A'$ such that one of its neighbors, say y , satisfies $y \in A'1B'$. Do: $B \Leftarrow y$ and $C \Leftarrow x$, which decreases the width.

ii. $x \in B' \deg A'$ and $y \in N(x)$ imply $y \in A'2B'$ for any x and y .

A. Suppose that there are distinct $x_1, x_2 \in B' \deg A'$ having intersecting neighborhoods. First we will show that we may assume $N(x_1) \neq N(x_2)$. Indeed, if $\deg(x_1) = \deg(x_2) = 1$, then we have $N(x_1) = N(x_2)$, and therefore $|A'| = |N(x_1)| = 1$ and $B' = \{x_1, x_2\}$, which contradicts $|A'| > |B'|$. Similarly, if $\deg(x_1) = \deg(x_2) = 2$, then we also have a contradiction with $|A'| > |B'|$. So let $\deg(x_1) = \deg(x_2) = 3$ and $N(x_1) = N(x_2)$. First we do an exchange $B \Leftarrow z$ and $C \Leftarrow x_2$ for any $z \in C3A$. Then we have $N(x) \neq N(z)$, since otherwise $N(x_1) = N(x_2) = N(z)$ would force $G = K_{3,3}$, which is excluded by the assumption of the theorem. Therefore, we have $N(x_1) \neq N(x_2)$, as claimed. Choose any $w \in C \deg A$. Now, let $u \in N(x_1) \cap N(x_2)$.

- If $u \notin N(w)$ then do: $B \Leftarrow u$, $B \Leftarrow w$ and $C \Leftarrow \{x_1, x_2\}$.
 - If $u \in N(w)$ but there is a third $x_3 \in B' \deg A'$, then u is not adjacent to x_3 so do: $B \Leftarrow w$, $C \Leftarrow u$, and $C \Leftarrow x_3$.
 - $u \in N(w)$ and every vertex in $N(x_1) \cap N(x_2)$ is adjacent to w and $B' \deg A' = \{x_1, x_2\}$ or no two distinct $x_1, x_2 \in B' \deg A'$ have intersecting neighborhoods. In this case we delete $B' \deg A'$ and all those vertices which are adjacent to the two vertices of $B' \deg A'$ from $G(A', B')$. Since we have assumed earlier that $x \in B' \deg A'$ and $y \in N(x)$ imply $y \in A'2B'$, for any x and y , so each of the remaining vertices will have degree 1 or 2. Hence the resulting graph $G'(A', B')$ decomposes into maximal paths. If all paths have initial and terminal vertices in different partitions then $|B'| = |A'|$, which is a contradiction. Without loss of generality, let $u_0, v_1, u_2, v_3, \dots, u_{2m}$ be a maximal path of $G'(A', B')$ such that u_0 is adjacent to $x_1 \in B' \deg A'$ and $u_{2m} \in A'$. If it is not the case that $u_{2m} \in N(x_2)$ for some $x_2 \in B' \deg A'$ and $x_2 \neq x_1$, then $C \Leftarrow x_1$ and $A'' \Leftrightarrow B''$, where $A'' = \{u_0, u_2, \dots, u_{2m}\} \subseteq A'$ and $B'' = \{v_1, v_3, \dots, v_{2m-1}\} \subseteq B'$. Suppose, on the other hand, that $u_0 \in N(x_1) - N(x_2)$ and $u_{2m} \in N(x_2) - N(x_1)$ for distinct $x_1, x_2 \in B' \deg A'$. If w is adjacent to some $v \in A''$, then v is not adjacent to at least one of x_1 and x_2 , say x_2 . Hence we do: $B \Leftarrow w$, $C \Leftarrow \{v, x_2\}$ (see Fig. 1). Otherwise, w is independent of A'' . Then we do: $B \Leftarrow w$, $C \Leftarrow \{x_1, x_2\}$, $A'' \Leftrightarrow B''$.
- B. There is only one $x \in B' \deg A'$ and $y \in N(x)$ implies $y \in A'2B'$ for any y . Since $|A'| > |B'|$, the only possibility is that $|A'| = |B'| + 1$. Do: $A' \Leftrightarrow B'$.

We finally conclude that the width has been decreased in all the cases and that the proof is complete. \square

It is easy to see that a single step of decreasing the color width can be done in linear time. Since such a decreasing procedure must be applied at most $n/3 - 1$ times, we have $O(n^2)$ as the computational complexity of the whole equalizing procedure.

3 NP-hardness of the problem

Let us consider the following combinatorial decision problem:

IS(G, l): given a subcubic graph G on n vertices and an integer l ; the question is whether G has an independent set I of size at least l ,

and its subproblem IS($G, .35n$), i.e. IS(G, l), where $l = 35n/100$ and n is divisible by 20, in symbols $20|n$.

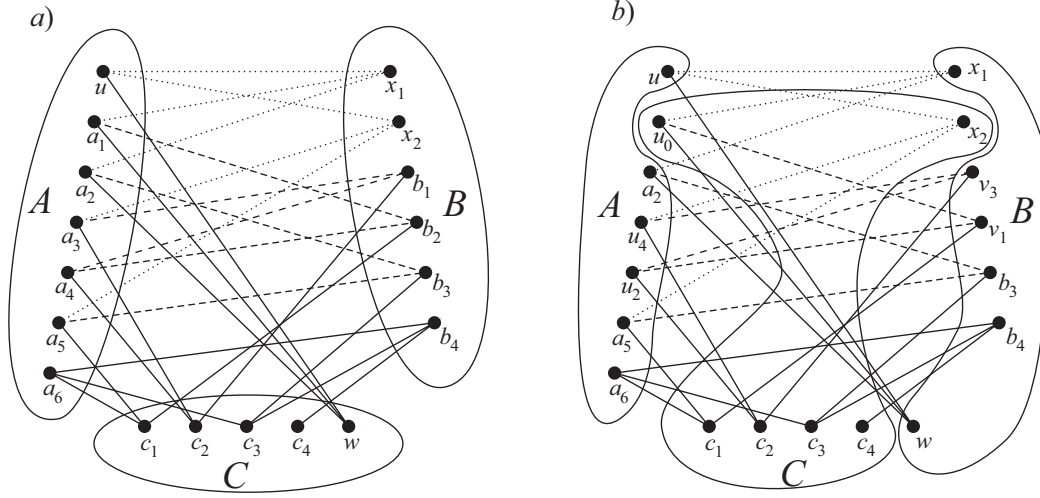


Figure 1: An example of a subcubic graph fulfilling the assumption of Case 5(b)ii.A. a) $G(A', B')$ consists of $A' = \{u, a_1, \dots, a_5\}$, $B' = \{x_1, x_2, b_1, \dots, b_3\}$ and dashed and dotted edges. Graph $G''(A', B')$ consists of vertices $A'' = \{a_1, \dots, a_5\}$, $B'' = \{b_1, \dots, b_3\}$, and edges drawn with a dashed line. We choose as the maximal path mentioned in the proof of Theorem 2.2 path a_1, b_2, a_4, b_1, a_3 . We rename these vertices as u_0, v_1, u_2, v_3, u_4 - Fig. b). Here $u_0 \in N(x_1) \setminus N(x_2)$ and $u_{2m} = u_4 \in N(x_2) \setminus N(x_1)$. Since $w \in N(u_0)$, we do: $B \Leftarrow w$, $C \Leftarrow \{u_0, x_2\}$ - the final result in Fig. b).

Note that the $\text{IS}(G, l)$ problem is NP-complete even if G is cubic [8] and remains so if $20|n$. This is so because we can enlarge a cubic graph G by adding to it a bipartite cubic graph on j vertices, where $j = 6, 8, 10, 12, 14, 16, 18, 22$, or 24 , so that the number of vertices in the new graph is divisible by 20. It is easy to see to see that G has an independent set of size at least l if and only if the new graph has an independent set of size at least $l + j/2$.

In [5] we proved that the problem of deciding whether a cubic graph has a coloring of type $[4n/10, 3n/10, 3n/10]$ is NP-complete. In the following we strengthen and generalize this result to semi-equitable k -colorings ($k \geq 4$) and subcubic graphs.

Lemma 3.1. *Problem $\text{IS}(G, .35n)$ is NP-complete.*

Proof. Let G be an n -vertex cubic graph, where $20|n$. We ask whether there exists an independent set of size at least l . We reduce this question to another one about the existence an independent set of size at least $35n/100$ in a subcubic graph H , i.e. $\text{IS}(H, .35n)$. Let $r = \lfloor 35n/100 - l \rfloor$.

If $l \geq 35n/100$ then we construct a subcubic graph H by adding to G a graph $r(4K_4 \cup K_3 \cup K_1)$, which results in increasing the order of graph by $20r$. Each such subgraph on 20 vertices provides 6 new vertices in any independent set. Thus, if G has an independent set of size at least l then H has an independent set of size at least $35n/100$ and vice versa.

If $l < 35n/100$ then we add to G $2r$ graphs P , where P is the Petersen graph. This time each such subgraph on 20 vertices provides 8 new independent vertices.

Thus, if G has an independent set of size at least l then H has an independent set of size at least $35n/100$ and vice versa. \square

Lemma 3.2. *Let G be a subcubic graph such that $20|n$, where n is the number of vertices of G . The problem of deciding whether G has a semi-equitable coloring of type $[7n/20, 13n/60, 13n/60, 13n/60]$ is NP-complete.*

Proof. We prove that G has a coloring of type $[7n/20, 13n/60, 13n/60, 13n/60]$ if and only if there is an affirmative answer to $\text{IS}(G, .35n)$.

Suppose first that G has the above 4-coloring. Then the first color class of size $7n/20$ is an independent set that forms a solution to $\text{IS}(G, .35n)$.

Now suppose that there is a solution I to $\text{IS}(G, .35n)$. Thus $|I| \geq 7n/20$. Clearly, in this case there exists an independent set I' of size exactly $l = 7n/20$. From Theorem 2.2 we know that there is an equitable 3-coloring of the remaining subcubic graph $G - I'$. Thus G has a semi-equitable 4-coloring of type $[7n/20, 13n/60, 13n/60, 13n/60]$. \square

It is easy to see that Lemma 3.2 can be generalized to hold for any number $k \geq 4$ of colors. In other words, we have the following

Theorem 3.3. *Given an n -vertex subcubic graph, a constant $k \geq 4$, and an integer function $s = s(n)$. Finding a semi-equitable k -coloring of G of type $[s, \lceil \frac{n-s}{k-1} \rceil, \dots, \lfloor \frac{n-s}{k-1} \rfloor]$ is NP-hard, if $s \geq 7n/20$.*

4 Semi-equitable k -coloring of subcubic graphs

In Section 3 we have proved that the problem of semi-equitable k -coloring of n -vertex cubic graph, where the largest color class is of size $s(n)$ at least $7n/20$, is NP-hard for $k \geq 4$. It turns out that if we diminish the lower bound for $s(n)$ slightly, namely by the value of $n/60$, then we get an instance of the problem, which is polynomially solvable for any $k \geq 4$.

Theorem 4.1. *Given a connected n -vertex subcubic graph G , a constant $k \geq 4$, and an integer function $s = s(n)$. Finding a semi-equitable k -coloring of G of type $[s, \lceil \frac{n-s}{k-1} \rceil, \dots, \lfloor \frac{n-s}{k-1} \rfloor]$ is solvable in $O(n^2)$ time, if $s \leq \lceil n/3 \rceil$.*

Proof. First, we have to determine an independent vertex set I of size s , $s \leq \lceil n/3 \rceil$. We achieve this by assigning 3 colors to the vertices of subcubic graph G due to the linear time procedure given by Skulrattanakulchai in [10]. The largest color class is of size at least $\lceil n/3 \rceil$. By choosing s vertices of them arbitrarily, we get the set I . Now, we have to color graph $G - I$ equitably with $k - 1$ colors. If $k \geq 5$ then $(k - 1)$ -coloring of $G - I$ is guaranteed by Theorem 1.1. Such a coloring can be obtained in $O(n^2)$ time. If $k = 4$, $G - I$ can be properly colored with 3 colors, as a subcubic graph different from K_4 . Since $G - I$ is also different from $K_{3,3}$, we can apply an $O(n^2)$ -time procedure from the proof of Theorem 2.2 for equalizing a given 3-coloring of $G - I$. \square

One can ask about a semi-equitable 3-coloring of a subcubic graph G . The problem was discussed in [4], where we proved

Theorem 4.2 ([4]). *If G , n -vertex cubic graph, has an independent set I of size $|I| \geq 2n/5$, then it has a semi-equitable coloring of type $[|I|, \lceil \frac{n-|I|}{2} \rceil, \lfloor \frac{n-|I|}{2} \rfloor]$. \square*

Moreover, we noticed that a cubic graph usually has such a big independent set. This is so because Frieze and Suen [3] proved that for almost all cubic graphs G their independence number $\alpha(G)$ fulfills the inequality $\alpha(G) \geq 4.32n/10 - \epsilon n$ for any $\epsilon > 0$. In practice this means that a random cubic graph is very likely to have an independent set of size $k \geq 2n/5$ and the probability of this fact increases with n .

The above result holds also for almost all subcubic graphs.

Tables 1 and 2 gather the complexity status for semi-equitable k -coloring for cubic and subcubic graphs, respectively.

k	$s \leq \lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil < s < \lfloor \frac{2n}{5} \rfloor$	$\lfloor \frac{2n}{5} \rfloor \leq s < \frac{n}{2}$	$s = \frac{n}{2}$	$\frac{n}{2} < s$
3	-	?	NPH	$O(n)^*/-$	-
≥ 4	$O(n^2)$?	NPH	$O(n)^*/-$	-

Table 1: The complexity of semi-equitable k -coloring of cubic graphs. The "-" sign means that the corresponding solution does not exist. The "*" sign means that the solution concerns bipartite cubic graphs only.

k	$s \leq \lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil < s < \lfloor \frac{7n}{20} \rfloor$	$\lfloor \frac{7n}{20} \rfloor \leq s \leq \lfloor \frac{3n}{4} \rfloor$	$\lfloor \frac{3n}{4} \rfloor < s$
3	-	?	NPH	-*
≥ 4	$O(n^2)$?	NPH	-*

Table 2: The complexity of semi-equitable k -coloring of subcubic graphs. The "-" sign means that the corresponding solution does not exist. The "*" sign means that the corresponding negative result concerns connected graphs only.

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